Solution to Problem 18) Let A be a square matrix whose rows are orthonormal vectors. Then $AA^{*T} = I$, simply because each column of A^{*T} is the conjugate transpose of a row of A. This makes A^{*T} the inverse of A, that is, $A^{*T} = A^{-1}$. (In other words, A is unitary.) However, the inverse matrix has the property that $AA^{-1} = A^{-1}A = I$. Consequently, $A^{*T}A = I$. The later identity shows that the columns of A are orthonormal vectors as well.

As a special case, consider the 2×2 $ABCD$ matrix, whose rows $(A \quad B)$ and $(C \quad D)$ are assumed to be orthonormal. We thus have

$$|A|^2 + |B|^2 = |C|^2 + |D|^2 = 1, \tag{1a}$$

$$AC^* + BD^* = 0. \tag{1b}$$

The latter equation yields $A/B = -(D/C)^*$, that is,

$$(|A|/|B|)\exp[\mathrm{i}(\varphi_A - \varphi_B)] = (|D|/|C|)\exp[\mathrm{i}(\varphi_C - \varphi_D \pm \pi)]. \tag{2}$$

Given that Eq.(1a) may be written as $|B|^2[(|A|/|B|)^2 + 1] = |C|^2[1+(|D|/|C|)^2]$, substitution from Eq.(2) now reveals that $|B| = |C|$ and, consequently, that $|A| = |D|$. We also find from Eq.(2) that $\varphi_A + \varphi_D = \varphi_B + \varphi_C \pm \pi$.

Let us now consider the columns of the $ABCD$ matrix, whose orthogonality would require that $|A|^2 + |C|^2 = |B|^2 + |D|^2 = 1$ and $AB^* + CD^* = 0$. Since $|B| = |C|$, the first of these relations is equivalent to Eq.(1a). As for the second relation, its satisfaction would require that $(|A|/|C|)\exp[\mathrm{i}(\varphi_A - \varphi_C)] = (|D|/|B|)\exp[\mathrm{i}(\varphi_B - \varphi_D \pm \pi)]$. This, however, is guaranteed because it is already established that $|B| = |C|$, $|A| = |D|$, and $\varphi_A + \varphi_D = \varphi_B + \varphi_C \pm \pi$.